NUMBER AND ALGEBRA

1 NUMBER SYSTEMS:

1.1 Number System
1.2 Significant Figures
1.3 Scientific Notation
1.4 Rounding Errors
1.5 Computation Errors

Formulae:

- $\boxed{\text{Absolute Error} = |\text{ True value} - \text{Measurement }|}$

- $\boxed{\text{Relative Error} = \dfrac{\text{Absolute Error}}{\text{True Value}}}$

- $\boxed{\text{Percentage Error} = \text{Relative Error X } 100}$

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2 SEQUENCES AND SERIES:

2.1 Arithmetic Sequences & Series
2.2 Geometric Sequences & Series

Formulae:

- $\boxed{U_n = S_n - S_{n-1}}$ To find a specific term given the sum of that number of terms and the sum of all terms up to and not including that term.

- $\boxed{U_2 - U_1 = U_3 - U_2 = d}$ To test whether a sequence is arithmetic.

- $\boxed{\dfrac{U_2}{U_1} = \dfrac{U_3}{U_2} = r}$ To test whether a sequence is geometric.

- $\boxed{U_n = a + (n-1)d}$ To find the n^{th} term of an arithmetic sequence given the common difference (d) and first term (a).

- $\boxed{U_n = a\,r^{\,n-1}}$ To find the n^{th} term of a geometric sequence given the common ratio (r) and the first term (a).

- $\boxed{S_n = \dfrac{n}{2}[\,2a + (n-1)d]}$ To find the sum of n terms in an arithmetic series.

- $\boxed{S_n = \frac{n}{2}[a + l]}$ Same as above, but given the last and first terms.

- $\boxed{S_n = \frac{a(r^n - 1)}{r - 1}}$ OR $\boxed{S_n = \frac{a(1 - r^n)}{1 - r}}$ To find the sum of n terms in a geometric series.

- $\boxed{S_\infty = \frac{a}{1 - r} \text{ where } |r| < 1}$ To find the sum of an infinite (geometric) series.

- $\boxed{I = \frac{P\,r\,n}{100}}$ — Simple Interest formula where
  A = principal (amount) invested
  r = rate (7% —> r = 7)
  n = no. of time periods

- $\boxed{A = P\left(1 + \frac{r}{100}\right)^n}$ OR $\boxed{A = P\,R^n}$ — Compound interest formula. $\left(\mathbf{R} = 1 + \frac{r}{100}\right)$

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3 INDICES & LOGARITHMS:

3.1 Laws of Indices
3.2 Logarithms
3.3 Change of base and the natural base "e"

Formulae:

Index Laws	**Logarithm Laws**
$\boxed{a^m \times a^n = a^{m+n}}$	$\boxed{\log_a m + \log_a n = \log_a(mn)}$
$\boxed{a^m \div a^n = a^{m-n}}$	$\boxed{\log_a m - \log_a n = \log_a\left(\frac{m}{n}\right)}$
$\boxed{\left(a^m\right)^n = a^{mn}}$	$\boxed{\log_a x^y = y \times \log_a x}$
$\boxed{a^{-m} = \frac{1}{a^m}}$	$\boxed{\log_a \frac{1}{x} = \log_a x^{-1} = -\log_a x}$
$\boxed{a^0 = 1}$	$\boxed{\log_a 1 = 0}$
$\boxed{a^1 = a}$	$\boxed{\log_a a = 1}$

$\boxed{\log_b a = \dfrac{\log_m a}{\log_m b}}$ — Change of Base

Notes:

"e" is the natural exponential. e^1 is equal to 2.71828182845904523…
$\log_e$ is the natural logarithm (also called ln)

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4 THE BINOMIAL THEOREM:

4.1 The Binomial Theorem

$\boxed{(a+b)^n = \sum_{r=0}^{n} {}^nC_r\, a^{n-r} b^r}$ Binomial Theorem Formula

Example:

$(x + y)^4$
$= {}^4C_0x^4y^0 + {}^4C_1x^3y^1 + {}^4C_2x^2y^2 + {}^4C_3x^1y^3 + {}^4C_4x^0y^4$
$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

FUNCTIONS & EQUATIONS

1 GRAPHS:

1.1 Linear Equations & Graphs
1.2 Standard Forms of Linear Equations
1.3 Linear Simultaneous Equations
1.4 Plotting Graphs

Formulae:

$\boxed{\mathbf{d} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}$ — Distance formula, where (x1,y1) & (y1,y2) are 2 points on a line.

$\boxed{\mathbf{m} = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)}$ — Midpoint formula.

Notes:

• To find the axes intercepts on a graph, let y = 0, and x = 0, to find x & y respectively.
• Simultaneous Equations— Graphical, Elimination, and Substitution Methods.

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2 QUADRATIC FUNCTIONS & EQUATIONS:

2.1 Quadratic Equations and the discriminant
2.2 Quadratic Graphs
2.3 Simultaneous Equations

Formulae:

- $\boxed{\mathbf{a}x^2 + \mathbf{b}x + \mathbf{c} = 0}$ — General form of quadratic Equation

- $\boxed{x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$ — Quadratic Formula.

- $\boxed{y = a(x - h)^2 + k}$ Turning point form; graph is dilated by a factor of **a** vertically and is translated *h* units to the **right** and *k* units **upwards,** ie. the vertex is translated to point *(h, k)* from the origin.

Other:

- Solving Quadratics— Factorisation method, Completing Square, Quadratic Formula.
- The Discriminant $b^2 - 4ac$, >0 => 2 sol's, =0 => 1 sol'n, <0 => no sol's.
- Sketching quadratic graphs— Turning point form, intercept method.
- Quadratic simultaneous eq'ns.

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3 FUNCTIONS AND RELATIONS

3.1 Relations
3.2 Functions
3.3 Some Standard Functions (Abs. Value, Reciprocal & Exponential)
3.4 Composite Functions
3.5 Inverse Functions
3.6 Logarithmic Functions

Definition:

Function: A function is a relation in which no two of the ordered pairs have the same first element.
ie. {(0,2),(1,2),(2,1)} is a function, whereas {(0,2),(0.-2),(2,1)} is not.

Notes:

Function notation: 1. f:X |—> Y, where $y = f(x)$
2. f:X |—> Y, $y = f(x)$
3. $y = f(x), x \in X$

Formulae:

- $\boxed{f(x) = a^x \text{ where } a > 0}$

- $\boxed{f(x) = \log_a x \text{ where } a > 0 \ \& \ 0 < x < \infty}$

- $\boxed{f(x) = |ax + b|}$

- $\boxed{f_o g(x) = f(g(x))}$

- $\boxed{f_o f^{-1} = f(f^{-1}(x)) = x}$

Composite Functions:

Addition: $(f + g)(x) = f(x) + g(x)$
$\delta_{f+g} = \delta_f \cap \delta_g$

Subtraction: $(f - g)(x) = f(x) - g(x)$
$\delta_{f-g} = \delta_f \cap \delta_g$

Multiplication: $(f \times g)(x) = f(x) \times g(x)$
$\delta_{f\times g} = \delta_f \cap \delta_g$

Division: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$
$\delta_{\frac{f}{g}} = \delta_f \cap \delta_g \setminus \{x: g(x) = 0\}$

•NB: For $(g_o f)(x) = g(f(x))$ to exist, then $r_f \subseteq \delta_g$.

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4 TRANSFORMATIONS OF GRAPHS

4.1 Transformations (translations, reflections and dilations)

Formulae:

If $y = f(x)$, then:

- $y = f(-x)$ is a reflection about the y-axis
- $y = -f(x)$ is a reflection about the x-axis.
- $y = f(x - b)$ is a translation along the x-axis b units.
  $y = f(x) + b$ is a translation along the y-axis b units.

Definitions:

Even Functions: This is where $f(x) = f(-x)$. They are symmetrical about the y-axis. eg. $y = x^2$, $y = x^4 + 4$

Odd Functions: This is where $f(x) = -f(-x)$. They are invariant under rotation of 180º about the origin.
eg. $y = x^3$

CIRCULAR FUNCTIONS & TRIGONOMETRY

1 SOLUTION OF TRIANGLES

1.1 The trigonometric ratios
1.2 The sine & cosine rules
1.3 Areas
1.4 Proof of Sine & Cosine Rules

Formulae:

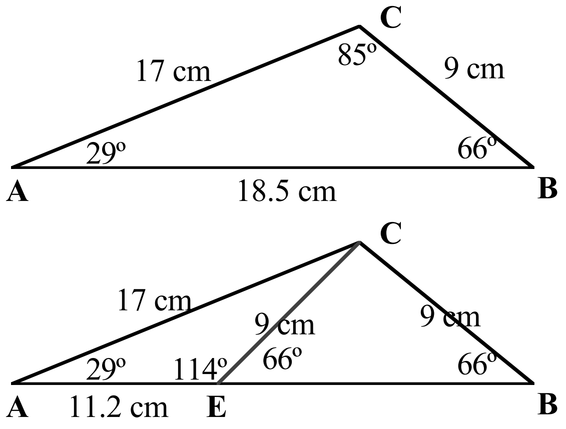
- $\boxed{\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}}$ — Sine Rule

- $\boxed{a^2 = b^2 + c^2 - 2bc\cos A}$ OR $\boxed{\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}}$ — Cosine Rule

- $\boxed{A = \dfrac{1}{2} b\,c \sin A}$ — Area of a triangle

Notes:

- Ambiguous Case of Sine Rule:
  — For the triangle below, there are two possible triangles, ABC and ACE:



2 RADIAN MEASURE

2.1 Arcs & Sectors

Formulae:

- $\boxed{s = r\theta^c}$ — To find the length of a sector
- $\boxed{A = \frac{1}{2}r^2\,\theta}$ — To find the area of a sector

Notes:

- Radians— 360º $= 2\pi$ radians.
  180º $= \pi$ radians.
  90º $= \frac{\pi}{2}$ radians.
  45º $= \frac{\pi}{4}$ radians.
  60º $= \frac{\pi}{3}$ radians.
  30º $= \frac{\pi}{6}$ radians.

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3 THE UNIT CIRCLE

3.1 Trigonometric Functions
3.2 Inverse Trigonometric Functions
3.3 Applications
3.4 Identities & Equations

Formulae:

- $\boxed{f(\mathrm{x}) = \mathrm{a}\sin\mathrm{b}(\mathrm{x} - \mathrm{e}) + \mathrm{c}}$
- $\boxed{f(\mathrm{x}) = \mathrm{a}\cos\mathrm{b}(\mathrm{x} - \mathrm{e}) + \mathrm{c}}$
- $\boxed{f(\mathrm{x}) = \mathrm{a}\tan\mathrm{b}(\mathrm{x} - \mathrm{e}) + \mathrm{c}}$
- $\boxed{\sin^2\theta + \cos^2\theta = 1}$
- $\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}}$

Notes:

- For the function $f(\mathrm{x}) = a\sin b(\mathrm{x} - \varepsilon) + c$:
  — a alters the amplitude of the curve.

— *b* alters the period (τ) of the curve ($\tau = \frac{2\pi}{n}$ for $f(x) = \sin(nx)$ or $\cos(nx)$ or $\frac{\pi}{n}$ for $f(x) = \tan(nx)$.

— ε translates the curve ε units to the right.

— *c* translates the curve *c* units upwards.

• Inverse Functions: $f(x) = \sin x$ etc. are taken over an interval so their inverse functions $f(x) = \arcsin x$ can be defined as a function:

ie. $f(x) = \text{Sin}\, x,\ x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, ie. $f(x) = \arcsin x,\ x \in [-1,1]$

$f(x) = \text{Cos}\, x,\ x \in [0, \pi]$, ie. $f(x) = \arccos x,\ x \in [-1,1]$

$f(x) = \text{Tan}\, x,\ x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ ie. $f(x) = \arctan x,\ x \in [-1,1]$

• Identities & Equations:

eg. Solve $2\cos^2 x - \sin x = 1,\ -2\pi \le x \le 2\pi$

$2\cos^2 x - \sin x = 1$

$2(1 - \sin^2 x) - \sin x - 1 = 0$

$2 - 2\sin^2 x - \sin x - 1 = 0$

$-2\sin^2 x - \sin x + 1 = 0$

$2\sin^2 x + \sin x - 1 = 0$

Let sin x = X

$\mathbf{2X^2 + X - 1 = 0}$

$\mathbf{(2X - 1)(X + 1) = 0}$

$(2\sin x - 1)(\sin x + 1) = 0$

$2\sin x - 1 = 0$

$\sin x = \frac{1}{2}$

$x = -330^\circ, -210^\circ, 30^\circ, 150^\circ$

$= -\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$

$\sin x + 1 = 0$

$\sin x = -1$

$x = -90^\circ, 270^\circ$

$= -\frac{\pi}{2}, \frac{3\pi}{2}$

$x = -\frac{11\pi}{6}, -\frac{7\pi}{6}, -\frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$ radians.

VECTOR GEOMETRY

1 VECTORS

1.1 Scalar and Vector Quantities
1.2 Vector Arithmetic
1.3 Unit Vectors
1.4 Scalar Product
1.5 Resultants
1.6 Representation of A Line in the Plane
1.7 Lines in Three Dimensions

Formulae:

- $\boxed{\left|a\vec{i} + b\vec{j}\right| = \sqrt{a^2 + b^2}}$ — Magnitude of a Vector
- $\boxed{\cos\theta = \dfrac{\vec{a} \bullet \vec{b}}{\left|\vec{a}\right|\left|\vec{b}\right|}}$ — Scalar Product
- $\boxed{\text{Resolute of } \vec{a} \text{ on } \vec{b} = \dfrac{\vec{a} \bullet \vec{b}}{\left|\vec{b}\right|^2}\vec{b}}$
- $\boxed{\boldsymbol{r} = \boldsymbol{p} + t\boldsymbol{d}}$ — Equation of a line.

Notes:

- Vectors can be represented in several ways:
  - By arrows
  - With notation, eg. AB
  - Using the $\vec{i}, \vec{j}, \vec{k}$ notation.
  - As a column vector, eg. $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.
- Vectors are added 'nose to tail'. We can add up the separate components of $\vec{i}, \vec{j}, \vec{k}$ or the separate lines in the column vector notation. To subtract a vector we simply add the negative of the vector when it is positive.

  eg. $\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix}$

  eg. $\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} - \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix}$

- Unit vectors are found by finding the magnitude of the vector, finding 1 over this, and multiplying this by the vector. It is of length one unit in the same direction as the vector itself:

  $\left\|\begin{pmatrix} -3 \\ 6 \\ 4 \end{pmatrix}\right\| = \sqrt{(-3)^2 + 6^2 + 4^2} = \sqrt{61}$, therefore the required vector is $\dfrac{1}{\sqrt{61}}\begin{pmatrix} -3 \\ 6 \\ 4 \end{pmatrix}$

- Scalar Product (or dot product) is found using the formula $\cos\theta = \dfrac{\vec{a} \bullet \vec{b}}{|\vec{a}||\vec{b}|}$. It is also found by multiplying the coefficients of the $\vec{i}, \vec{j}, \vec{k}$ notation. The most usual use of the scalar product is to calculate the angle between vectors.

  eg. The angle between $\begin{pmatrix} 0 \\ -5 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ -1 \\ -3 \end{pmatrix}$ is calculated as follows:

  $$\left|\begin{pmatrix} 0 \\ -5 \\ 4 \end{pmatrix}\right| = \sqrt{0^2 + (-5)^2 + 4^2} = \sqrt{41}$$

  $$\left|\begin{pmatrix} -5 \\ -1 \\ -3 \end{pmatrix}\right| = \sqrt{(-5)^2 + (-1)^2 + (-3)^2} = \sqrt{35}$$

  Scalar product:

  $$\begin{pmatrix} 0 \\ -5 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} -5 \\ -1 \\ -3 \end{pmatrix} = 0 \_ (-5) + (-5) \_ (-1) + 4 \_ (-3) = -7$$

  Finally we calculate the angle:

  $$\cos \_ = \frac{-7}{\sqrt{41} \text{ ¥ } \sqrt{35}} \text{, therefore } \_ \approx 101^\circ$$

  <u>Properties of the Scalar Product</u>:

  1. $\mathbf{v} \bullet \mathbf{w} = \mathbf{w} \bullet \mathbf{v}$
  2. $\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = \mathbf{u} \bullet \mathbf{v} + \mathbf{v} \bullet \mathbf{w}$
  3. $(k\mathbf{v}) \bullet \mathbf{w} = k(\mathbf{v} \bullet \mathbf{w})$
  4. $\mathbf{v} \bullet \mathbf{v} = |\mathbf{v}|^2$
  5. If $\mathbf{v} \perp \mathbf{w}$ and $\mathbf{v} \neq 0$, $\mathbf{w} \neq 0$, then $\mathbf{v} \bullet \mathbf{w} = 0$
  6. If $\mathbf{v} \parallel \mathbf{w}$ and $\mathbf{v} \neq 0$, $\mathbf{w} \neq 0$, then $\mathbf{v} \bullet \mathbf{w} = \pm |\mathbf{v}|\,|\mathbf{w}|$

- <u>Representation of a Line</u>:
  1. Graphically.
  2. Cartesian Form, eg. $x + y = 8$, or $y = 8 - x$.
  3. Parametric Form, $\mathbf{r} = \mathbf{p} + t\mathbf{d}$, where $\mathbf{p}$ is some point on the line, such as $\begin{pmatrix} 0 \\ 8 \end{pmatrix}$, and $\mathbf{d}$ is any vector on the line, such as $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, so:

     $$\mathbf{r} = \begin{pmatrix} 0 \\ 8 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

  4. Vector Form, $\mathbf{r} \bullet \mathbf{n} = D$, where $\mathbf{n}$ is a line opposite the line, with opposite gradient. A known point is substituted to find D, eg:

     For the line $x + y = 8$,

An opposite line is $\begin{pmatrix}1\\1\end{pmatrix}$, and a point on the line is $\begin{pmatrix}8\\0\end{pmatrix}$.

$D = 8$, therefore equation is $\mathbf{r} \bullet \begin{pmatrix}1\\1\end{pmatrix} = 8$.

A unit vector that is perpendicular to the curve can also be used in the form of the equation, $\mathbf{n} \bullet \mathbf{r} = d$, and in this case, d is the distance from the line to the origin.

For the line $x + y = 8$,

The unit vector is $\begin{pmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix}$, and a point is $\begin{pmatrix}0\\8\end{pmatrix}$.

Therefore the line is $\mathbf{r} \bullet \begin{pmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix} = 4\sqrt{2}$.

This can also apply to lines in 3 dimensions.

- The point-gradient equation, $y - y_1 = m(x - x_1)$, can be substituted into the parametric form of the line:
  $\mathbf{r} = \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x_1\\y_1\end{pmatrix} + t\begin{pmatrix}1\\m\end{pmatrix}$
- The Cartesian form of the line in the same way can be substituted into the vector form of the line:
  $\mathbf{r} \bullet \begin{pmatrix}x\\y\end{pmatrix}$, so in the case of $x + y = 8$, $\begin{pmatrix}x\\y\end{pmatrix} \bullet \begin{pmatrix}1\\1\end{pmatrix} = 8$.

STATISTICS AND PROBABILITY

1 STATISTICS

1.1 Frequency Diagrams
1.2 Statistical Measures
1.3 Measures of Spread
1.4 The Normal Distribution

Formulae:

- $\boxed{\overline{x} = \frac{\sum_{i=1}^{n} x_i}{n}}$ — Sample mean

- $\boxed{s_n = \sqrt{\dfrac{\sum_{i=1}^{n} f_i (x_i - x)^2}{n}}}$ — Sample Standard Deviation
- $\boxed{\dfrac{\sum_{i=1}^{n} f_i (x_i - \mu)^2}{n}}$ — Variance
- $\boxed{\sigma = \sqrt{\dfrac{\sum_{i=1}^{n} f_i (x_i - \mu)^2}{n}}}$ — Population Standard Deviation
- $\boxed{z = \dfrac{x - m}{\sigma}}$ — Translation of general normal curve to standard normal curve.

Notes:

Measures of Central Tendency:
- **Mode**— most frequent class of data. Scores can be bimodal, trimodal, etc.
- **Mean**— Numeric data is added, and divided by the number of items that we have. See formula above. Notation depends on whether we have a population or a sample set.
- **Median**— found by arranging all the data in order of size and selecting the middle item.

Measures of Spread:
- **Variance** & **Standard Deviation**— show how much the scores deviate from the mean score. Standard Deviation is the square root of the variance.
- **Quartiles** & **Inter-Quartile Range**— the 1st and 3rd Quartiles are the data items 1/4 and 3/4 way through some scores. (The 2nd Quartile is the median). The Inter-Quartile Range is the difference between the 3rd and 1st quartile, and is a measure of the spread of the scores.

The Normal Distribution:

- We use z-scores to determine the proportions of data.
- $p(z<a)$, where $a \geq 0$— we use the statistical tables.
- $p(z>a)$, where $a \geq 0$— we use $1 - P(z<a)$
- $p(z>a)$, where $a<0$— we use the symmetry of the graph, so:
  $$P(z>-b) = P(z<b)$$
- $p(z<a)$, where $a<0$— we use the symmetry of the graph, so:
  $$P(z<-b) = P(z>b) = 1 - P(z<b)$$
- For non-standard problems, we use the formula to revert the scores back to a normal distribution.

2 PROBABILITY

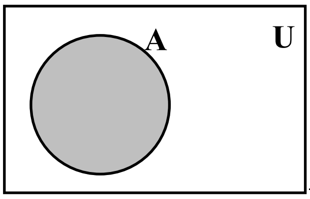
2.1 Probability
2.2 Probability and Venn Diagrams
2.3. Conditional Probability

- $\boxed{\lim_{N\to\infty} \frac{n(A)}{N} = p(A)}$ — Probability as a long-term frequency
- $\boxed{p(A) = \frac{n(A)}{n(U)}}$ — Probability of an event, in terms of:
  - Number of outcomes in which A occurs, and;
  - Total number of outcomes.
- $\boxed{p(A') = 1 - p(A)}$ — Probability of an event not occurring.
- $\boxed{p(A \cup B) = p(A) + p(B) - p(A \cap B)}$ — Probability of intersection.
- $\boxed{p(A \cap B) = p(A) \times p(B)}$ — Probability of union (both events happening).
- $\boxed{p(A|B) = \frac{p(A \cap B)}{p(B)}, p(B) \neq 0}$ — Conditional probability (A given B)

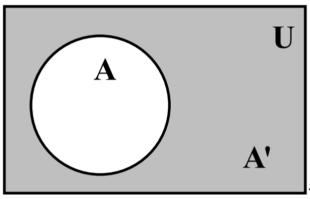
Therefore:

$$\boxed{\begin{aligned}
p(A'|B) &= \frac{p(A' \cap B)}{p(B)} = \frac{p(A) - p(A \cap B)}{p(B)} \\
p(A|B') &= \frac{p(A \cap B')}{p(B')} = \frac{p(A) - p(A \cap B)}{1 - p(B)} \\
p(B'|A) &= \frac{p(B' \cap A)}{p(A)} = \frac{p(A) - p(A \cap B)}{p(A)} \\
p(B|A') &= \frac{p(B \cap A')}{p(A')} = \frac{p(B) - p(A \cap B)}{1 - p(A)}
\end{aligned}}$$

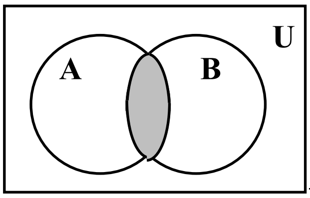
<u>The Venn Diagram:</u>



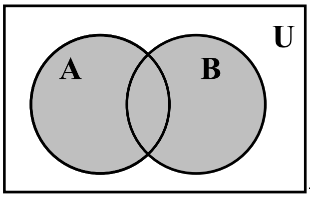
— Venn diagram, with sample space U and event A as subset.



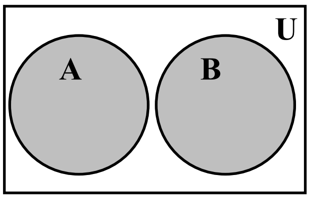
— The complement of A (ie. not A), denoted A'.

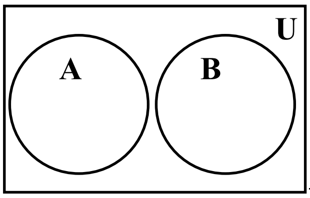


— The intersection of A & B, written A _ B



— The union of A & B, written A _ B.





— If (A _ B) = _ then A& B are mutually exclusive events, and the sets are called disjoint.

<u>Mutually Exclusive Events:</u>

- $(A \cap B) = \varnothing$
- $p(A \cap B) = 0$
- $p(A \cup B) = p(A) + p(B)$

<u>Independent Events:</u>

- $p(A|B) = p(A)$
- $p(B|A) = p(B)$
- $p(A \cap B) = p(A) \times p(B)$

CALCULUS

1 RATES OF CHANGE

1.1 Rates of Change
1.2 Qualitative Aspects of Change
1.3 From Average Rate of Change to An Instantaneous Rate of Change
1.4 The Process of Differentiation

Formulae:

- $\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$ — The derivative using first principles.

Notes:

- Finding the derivative to begin with starts with an average rate of change, then an instantaneous rate of change as we "zoom in", eg.
  From x = 1 to x = 2, then
  From x = 1 to x = 1.5, then
  From x = 1, to x = 1.1, then
  From x = 1, to x = 1.05, etc.
- Eventually using the limiting argument we can find the instantaneous rate of change, which is the derivative.

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2 DIFFERENTIATION

2.1 Differentiation
2.2 Derivative of Trigonometric, Exponential, and Logarithmic Functions

Formulae:

- $\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$ — Chain Rule
- $\boxed{\frac{d}{dx}(u \times v) = u'v + v'u}$ — Product Rule
- $\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - v'u}{v^2}}$ — Quotient Rule

Principle of Differentiation:

If $f(x) = a\,x^n$, then $f'(x) = a\,n\,x^{n-1}$.

Common Derivatives:

- $y = \sin(kx)$ _ $\frac{d}{dx} = k \cos(kx)$
- $y = \cos(kx)$ _ $\frac{d}{dx} = -k \sin(kx)$
- $y = \tan(kx)$ _ $\frac{d}{dx} = k \sec^2(kx)$

- $y = e^{kx} - \frac{d}{dx} = k\, e^{kx}$
- $y = \log_e(kx) - \frac{d}{dx} = \frac{1}{x}$

3 APPLICATIONS OF DIFFERENTIATION

3.1 Rates of Change
3.2 Tangents and Normals
3.3 Curve Sketching
3.4 Applied Maximum and Minimum Problems
3.5 Kinematics and Calculus

Formulae:

- $\boxed{\mathrm{P(x)} = \mathrm{R(x)} - \mathrm{C(x)} \Rightarrow \frac{d\mathrm{C}}{d\mathrm{x}} \,\&\, \frac{d\mathrm{R}}{dx} \,\&\, \frac{d\mathrm{P}}{dx}}$ — Economics— Profit, Revenue & Cost. The derivatives are the Marginal Profit, Revenue and Cost.
- $\boxed{y - y_1 = f'(x_1)(x - x_1)}$ — Equation of Tangent at (x_1, y_1).
- $\boxed{y - y_1 = -\frac{1}{f'(x_1)}(x - x_1)}$ — Equation of normal at (x_1, y_1).
- $\boxed{v = \frac{ds}{dt} \,\&\, a = \frac{dv}{dt}}$ — Physics— Displacement, Velocity & Acceleration.

Notes:

- Stationary points occur where $\frac{dy}{dx} = 0$.
- **Local maximum** occurs with positive values before and negative values after the point, also $f''(\mathrm{x}) < 0$ at that point.
- **Local minimum** occurs with negative values before and positive values after the point, also $f''(\mathrm{x}) > 0$ at that point.
- **Point of inflection** occurs with negative values before and after the point, or positive values both before and after the point. $f''(\mathrm{x}) = 0$ at that point if there is a point of inflection.

4 INTEGRATION

4.1 Integration
4.2 Solving for “c”
4.3 Standard Integrals
4.4 The Definite Integral
4.5 Applications of Integration

Formulae:

- $\boxed{\int f(x).dx = F(x) + c \text{ where } c \in \Re}$
- $\boxed{\text{If } y = ax^n \text{ then } \int ax^n.dx = \frac{ax^{n+1}}{n+1} + c}$ — Anti-differentiation & Integration.
- $\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$ — The Definite Integral.
- $\boxed{\text{Area} = \int_a^b f(x)\,dx}$ — Area and the Definite Integral.

Standard Integrals:

- $f(x) = \frac{1}{x} \Rightarrow \int f(x).dx = \log_e x + c.$
- $f(x) = \sin(kx) \Rightarrow \int f(x).dx = -\frac{1}{k} \cos(kx) + c.$
- $f(x) = \cos(kx) \Rightarrow \int f(x).dx = \frac{1}{k} \sin(kx) + c.$
- $f(x) = \sec^2(kx) \Rightarrow \int f(x).dx = -\frac{1}{k} \cos(kx) + c.$
- $f(x) = e^{kx} \Rightarrow \int f(x).dx = \frac{1}{k} e^{kx} + c.$

General Power Rule:

- If $\int f(x).dx = F(x) + c$, then $\int f(ax+b).dx = \frac{1}{a} F(ax+b) + c.$

Notes:

- Area = Integral of Curve. Applies to Kinematics:
  Area under Acceleration/Time curve = Velocity.
  Area under Velocity/Time curve = Displacement.

FURTHER CALCULUS (OPTION):

1 DIFFERENTIATION AND ITS APPLICATIONS

1.1 Further Differentiation
1.2 Further Optimisation Problems
1.3 Application of the Second Derivative
1.4 Sketching Rational Functions

Formulae:

- $\boxed{\dfrac{d}{dx}(a^{kx}) = k\,(\log_e a)a^{kx}}$
- $\boxed{\dfrac{d}{dx}(\log_a x) = \dfrac{1}{(\log_e a)x}}$
- $\boxed{\dfrac{d}{dx}(\tan kx) = \dfrac{k}{\cos^2 x} = k\sec^2 x}$

Notes:

- Further Optimisation Problems— End points can be local maxima and minima.
- The Second Derivative can be used to find local maxima and minima, eg.
    - If $f'(a) = 0$, and $f''(a) < 0$, there is a local maximum at $(a, f(a))$, etc.

- To sketch rational functions:
    - Find x & y-intercepts using $y = 0$, $x = 0$ respectively.
    - Find f '(x), and thence f '(x) = 0— hence determine turning points.
    - Find f "(x), and thence determine the nature of turning points, and any possible points of inflection.

- Graphs of the form $f(x) = \dfrac{ax + b}{cx + d}$ — have asymptotes (horizontal and vertical).
    - We use limiting arguments to find these asymptotes.

    eg.

    Considering the function $\dfrac{3x+1}{2x+4}$,

    When 2x + 4 =0, ie. x = 2, the function is undefined:

    $$\left.\begin{array}{l} \lim\left(\dfrac{3x+1}{2x+4}\right) \to +\infty \\ \lim\left(\dfrac{3x+1}{2x+4}\right) \to -\infty \end{array}\right]$$ — Hence there is a vertical asymptote at x =2.

    Also,

    $$\frac{3x+1}{2x+4} = \frac{3+\frac{1}{x}}{2+\frac{4}{x}}$$

    So,

$$\lim\left(\frac{3x+1}{2x+4}\right) = \lim\left(\frac{3+\frac{1}{x}}{2+\frac{4}{x}}\right) = \frac{3+0}{2+0} = \frac{3}{2}$$

$$\lim\left(\frac{3x+1}{2x+4}\right) = \lim\left(\frac{3+\frac{1}{x}}{2+\frac{4}{x}}\right) = \frac{3+0}{2+0} = \frac{3}{2}$$

Therefore, horizontal asymptote at $y = \frac{3}{2}$

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2 APPROXIMATIONS

2.1 Iteration
2.2 Approximate Solutions of Equations
2.3 Newton-Raphson Method

Formulae:

- $\boxed{\text{Iteration Function} \Rightarrow x_{n+1} = g(x_n)}$
- $\boxed{\text{Approximations Using Calculus—} f(x+h) \approx f(x) + h \times f'(x)}$
- $\boxed{\text{Newton–Raphson Method} \Rightarrow x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$

Notes:

There are several main ways to determine the approximate solutions to equations:

1. Method of Iteration:

$\cos x + 1 - 5x = 0$— we are finding the approximation to an accuracy of 0.0001

Therefore,

$$x = \frac{1+\cos x}{5}$$

$$g(x) = \frac{1+\cos x}{5}$$

Let us take our first approximation as x = 1.

$x_1 = 1$

$$x_2 = \frac{1+\cos(x_1)}{5}$$

$= \frac{1 + \cos(1)}{5}$
$= 0.30806$

$x_3 = \frac{1 + \cos(0.30806)}{5}$
$= 0.39058$

$x_4 = \frac{1 + \cos(0.39058)}{5}$
$= 0.38493$

$x_5 = 0.38536$
$x_6 = 0.38533$

The difference between x_5 and x_6 is 0.00003, so we can say that:
The solution to $\cos x + 1 - 5x = 0$ is $x = 0.3853$

When we solve equations this way, we are actually using the cobweb diagram, which converges into a solution.

2. Another Iterative Method Using Calculus:

$x^3 + x^2 + x - 1 = 0.$

We take $x_1 = 1$

$f(x + h) \approx f(x) + h\_f'(x)$

Therefore $f(1 + h) \approx f(1) + h\_f'(1)$

$f(x) = x^3 + x^2 + x - 1, f(1) = 2$

$f'(x) = 3x^2 + 2x + 1, f'(1) = 6.$

We want $f(x + h) = 0$.
Therefore,

$f(1) + h.f'(1) = 0$
$2 + 6h = 0$

Therefore, $h = -\frac{1}{3}$
This tells us that
$1 + \left(-\frac{1}{3}\right) = \frac{2}{3}$ is a better approximation.

$x_2 = \frac{2}{3}$

Repeating the process,

$x_3 = \frac{5}{9} \approx 0.556$

The actual approximation is 0.543 (3 d.p.).

3. Newton–Raphson Method:

Here we use the rule $\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$

Here we use $x^3 + x^2 + x - 1 = 0$, with $x_1 = 1$.

$$
\begin{aligned}
x_1 &= 1 \\
x_2 &= 1 - \frac{(1)^3 + (1)^2 + (1) - 1}{3(1)^2 + 2(1) + 1} \\
&= 1 - \frac{2}{6} \\
&= \frac{2}{3} \text{ or } 0.666 \\
x_3 &= \frac{2}{3} - \frac{(\frac{2}{3})^3 + (\frac{2}{3})^2 + (\frac{2}{3}) - 1}{3(\frac{2}{3})^2 + 2(\frac{2}{3}) + 1} \\
&= \frac{5}{9} \text{ or } 0.555 \\
x_4 &= 0.5438 \text{ (4 d.p.)}
\end{aligned}
$$

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3 INTEGRATION

3.1 Integration by Substitution
3.2 Applications (Integration By Substitution)
3.3 The Definite Integral
3.4 Further Applications
3.5 Numerical Approximation

Integration By Substitution:

This is where we substitute a variable (say *u*) to define a certain part of an integration and therefore simplify the problem. This is like the chain rule reversed. It is used as followed:

1. Define u (u is a function of the variable (often x) which is part of the integrand).
2. Convert the integrand from an expression in x to an expression in u.
3. Integrate and rewrite in terms of x.

eg. Find $\int (2x + 1)^4\, dx$

Let $u = 2x + 1$

$\frac{du}{dx} = 2$

$dx = \frac{1}{2}\, du$.

Therefore $\int (2x + 1)^4\, dx$

$= \int u^4 \times \frac{1}{2}\, du$

$= \frac{1}{2}\int u^4\, du$

$= \frac{1}{2} \times \frac{u^5}{5} + c$

$= \frac{(2x + 1)^5}{10} + c$

eg. 2. Find $\int x\sqrt{x - 1}\, dx$

Let $u = x - 1$

$\frac{du}{dx} = 1$

$du = dx$

However, we end up with $\int x\sqrt{x - 1}\, dx = \int x\sqrt{u}\, du$, with two variables, so we must also substitute for x.

Here, $x = u - 1$

Therefore $\int x\sqrt{x - 1}\, dx$

$= \int x\sqrt{u}\, du$

$= \int (u - 1)u^{1/2}\, du$

$= \int (u^{3/2} - u^{1/2})\, du$

$= \frac{2u^{5/2}}{5} - \frac{2u^{3/2}}{3} + c$

$$= \frac{2\sqrt{(x-1)^5}}{5} - \frac{2\sqrt{(x-1)^3}}{3} + c.$$

When this is applied to the definite integral the limits also change:

eg. Evaluate the definite integral $\displaystyle\int_{-2}^{1} \frac{4x}{\sqrt{x^2+4}}\,dx$

Let $u = x^2 + 4$

$\dfrac{du}{dx} = 2x$

We also need to change the limits to u.

For $x = -2$, $u = (-2)^2 + 4 = 8$

For $x = 1$, $u = (1)^2 + 4 = 5$

Therefore, $\displaystyle\int_{x=-2}^{x=1} \frac{4x}{\sqrt{x^2+4}}\,dx = \int_{u=5}^{u=8} \frac{4x}{\sqrt{u}}\frac{du}{2x}$

$$= 2\int_{8}^{5} u^{1/2}\,du$$

$$= 4\,[u^{1/2}]_8^5$$

$$= 4(\sqrt{5} - \sqrt{8})$$

<u>Area Between Two Curves:</u>

To determine the area between two curves, we must do the following:

1. Draw the graphs and see which curve lies on top of the other.
2. If no limits are given, find the points of intersection.
3. Integrate the difference between the two curves given by the limits set or determined as the points of intersection.

The formula used is thus:

$$\boxed{A = \int_a^b g(x)\,dx - \int_a^b f(x)\,dx = \int_a^b (g(x) - f(x))\,dx}$$

eg. Find the area enclosed by $g(x) = x + 2$, $f(x) = x^2 + x - 2$ and the lines $x = -1$ and $x = 1$:

We firstly determine that $g(x) \geq f(x)$ at $[-1,1]$.

Therefore,

$$\int_{-1}^{1}((x+2)-(x^2+x-2))\,dx = \int_{-1}^{1}(4-x^2)\,dx$$

$$= \left[4x-\frac{x^3}{3}\right]_{-1}^{1}$$

$$= \left(4-\frac{1}{3}\right)-\left(-4+\frac{1}{3}\right)$$

$$= \frac{22}{3} \text{ sq. units.}$$

eg. 2. Find the areas enclosed by $g(x) = x + 2, f(x) = x^2 + x - 2$

As shown before, $g(x) \geq f(x)$.
The points of intersection are $x = -2$ and $x = 2$.

Therefore,

$$\int_{-2}^{2}((x+2)-(x^2+x-2))\,dx = \int_{-2}^{2}(4-x^2)\,dx$$

$$= \left[4x-\frac{x^3}{3}\right]_{-2}^{2}$$

$$= \left(8-\frac{1}{3}\right)-\left(-8+\frac{8}{3}\right)$$

$$= \frac{32}{3} \text{ sq. units.}$$

Numerical Approximation:

Some functions do not have an indefinite integral, and it may not be possible to evaluate the indefinite integral exactly, so here we use an approximation with a table of values. One way to do this is with the trapezoidal rule.

The trapezoidal rule is defined as:

$$\boxed{\int_a^b f(x)\,dx \approx \frac{h}{2}(y_0 + 2(y_1 + y_2 + \ldots + y_{n-1}) + y_n)}$$

eg. 1. Evaluate $\displaystyle\int_0^1 \frac{1}{x+1}\,dx$ using four trapezia of equal length:

The interval [0,1] is broken into four intervals of equal length, which means that each is $\frac{1}{4}$ unit in length. We then construct a table of values:

	x_0	x_1	x_2	x_3	x_4
x	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$f(x) = \frac{1}{x+1}$	1	$\frac{4}{5}$	$\frac{2}{3}$	$\frac{4}{7}$	$\frac{1}{2}$

Therefore, $\displaystyle\int_0^1 \frac{1}{x+1}\,dx$

$$\approx \frac{\frac{1}{4}}{2}\left(1 + 2\left(\frac{4}{5} + \frac{2}{3} + \frac{4}{7}\right) + \frac{1}{2}\right)$$

≈ 0.6970

The real answer is $[\log_e(x+1)]_0^1 = \log_e 2 = 0.6931$

Absolute error = 0.6970 – 0.6931 = 0.0039